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A PROBLEM OF P. SEYMOUR ON NONBINARY MATROIDS[1]

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The following statement for $k=1, 2, 3$ has been proved by Tutte [4], Bixby [1] and Seymour [3] respectively: If M is a k-connected non-binary matroid and X a set of $k-1$ elements of M, then X is contained in some U_4^2 minor of M. Seymour [3] asks whether this statement remains true for $k=4$; the purpose of this note is to show that it does not and to suggest some possible alternatives.

1. Introduction and notation

The following statement for $k=1, 2, 3$, has been proved by Tutte [4], Bixby [1] and Seymour [3] respectively (definitions will be given momentarily):

($*$) *If M is a k-connected non-binary matroid and X a set of $k-1$ elements of M, then X is contained in some U_4^2 minor of M.*

Seymour [3] asks whether ($*$) also holds for $k=4$; the purpose of this note is to show that it does not (section 2) and to suggest some possible alternatives (section 3).

We do assume some familiarity with matroids. Our notation is that of [6], except that: we use r rather than ϱ for rank; we write $M\setminus X$ and M/X respectively for the matroids obtained from M by deleting and contracting the subset X of E, and we shorten these to $M\setminus x$ and M/x when $X=\{x\}$.

Throughout M will be a matroid of rank r on the set E. For subsets X, Y of E, we define

$$\delta(X, Y) := r(X)+r(Y)-r(X\cup Y),$$

a quantity which may be thought of as the rank of the "ideal" intersection of X and Y. Then recall that a *k-separation* of M is just a partition (X, Y) of E satisfying $|X|, |Y|\geq k$ and $\delta(x, y)\leq k-1$, and that M is said to be *k-connected* if it has no k' separation for any $k'<k$. Finally recall that U_n^k is the matroid on n elements whose independent sets are just those sets of size at most k.

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2. An example

If C is a circuit-hyperplane (i.e. a circuit and a hyperplane) of M, then there is also a matroid M' whose independent sets are just the independent sets of M plus the set C. We will say that M' is obtained from M by *relaxing* C. For instance if M is the (binary) matroid on the set $\{a, b, c, d\}$ with circuits $\{a, b\}$, $\{a, c, d\}$, $\{b, c, d\}$, then U_4^2 is obtained from M by relaxing the circuit $\{a, b\}$. (The relaxing construction appears frequently in the matroid literature, for instance in the proof of [2], or in [5], where the "whirls" are just relaxations of the "wheels".)

The example we wish to describe is a relaxation M' of a binary matroid M. This has the advantage of allowing us to draw conclusions regarding M' from an analysis of M; to which end we record a few simple (and surely well known) observations for which I know of no convenient reference.

We assume in what follows that M' is obtained from M by relaxing the circuit-hyperplane C. The following are easily verified.

(1) *The dual* $(M')^*$ *of* M' *is obtained from* M^* *by relaxing* $E\setminus C$.

(2) *If* $z\in C$, *then* (a) $M'\setminus z=M\setminus z$, *and* (b) $C\setminus\{z\}$ *is a circuit-hyperplane of* M/z *whose relaxation is* M'/z.

And dually:

(3) *If* $z\in E\setminus C$, *then* (a) $M'/z=M/z$, *and* (b) C *is a circuit-hyperplane of* $M\setminus z$ *whose relaxation is* $M'\setminus z$.

(4) *If* M *is binary, then any* U_4^2 *minor of* M' *meets each of* $C, E\setminus C$ *exactly twice.*

Proof. Suppose $N=M'\setminus Z_1/Z_2$ is such a minor. Then from (2) and (3) and the fact that M has no U_4^2 minor, we conclude that $Z_1\subseteq E\setminus C$ and $Z_2\subseteq C$. Moreover, we cannot have $|C\setminus Z_2|=1$, since the single element of $C\setminus Z_2$ would then be a loop of N; and we cannot have $C=Z_2$, since then for $z\in Z_2$ we would have $N= =M'\setminus(Z_1\cup\{z\})/(Z_2\setminus\{z\})$, which contradicts our first assertion. We thus have $|C\setminus Z_2|\geqq 2$, and dually $|E\setminus C\setminus Z_1|\geqq 2$, completing the proof. ∎

(5) *If* M *is connected then* M' *is nonbinary.*

Proof. The connection of M guarantees us a basis $\{c, d, z_1, \dots, z_{r-2}\}$ of M whose intersection with $E\setminus C$ is $\{c, d\}$. Taking $C=\{a, b, z_1, \dots, z_{r-2}\}$, we find that $N:=M\setminus(E\setminus C\setminus\{c, d\})/\{z_1, \dots, z_{r-2}\}$ is the matroid on $\{a, b, c, d\}$ described earlier, and that (see (2b), (3b)) the minor $N':=M'\setminus(E\setminus C\setminus\{c, d\})/\{z_1, \dots, z_{r-2}\}$ of M' is obtained from N by relaxing $\{a, b\}$. Thus $N'\cong U_4^2$ and M' is nonbinary. ∎

(6) *The connectivity of* M' *is at least that of* M.

Proof. For any $X\subseteq E$, $r_{M'}(X)\geqq r_M(X)$. ∎

It follows from the last three results that if we can find M and C with $|C|\geqq 3$ and M binary and 4-connected, then M' will be a 4-connected nonbinary matroid with the property that for any 3-element set $X\subseteq C$ (or $\subseteq E\setminus C$), M' has no U_4^2 minor containing X (and we will thus have disproved (∗) when $k=4$). It turns out that the required conditions on M and C are met when M is the binary matroid on the following fifteen vectors and C consists of the first seven of these vectors.

$$
\begin{array}{l}
(1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0)\\
(0,\ 1,\ 0\ \ 0,\ 0,\ 0,\ 0)\\
(0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0)\\
(0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0)\\
(0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0)\\
(0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0)\\
(1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0)\\
(1,\ 1,\ 0,\ 1,\ 0,\ 0,\ 1)\\
(1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1)\\
(0,\ 1,\ 1,\ 0,\ 0,\ 1,\ 1)\\
(0,\ 0,\ 0,\ 1,\ 1,\ 1,\ 1)\\
(0,\ 1,\ 1,\ 1,\ 1,\ 0,\ 1)\\
(1,\ 0,\ 1,\ 1,\ 0,\ 1,\ 1)\\
(1,\ 1,\ 0,\ 0,\ 1,\ 1,\ 1)\\
(0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1)
\end{array}
$$

Of the properties we require of M and C, only the 4-connection of M is not immediate. Of course checking 4-connection is a finite problem, but we sketch a few points which simplify the process. Most of the work (and it really is not overly time consuming if one exploits the symmetries of M) involves verifying the following.

(a) *Each circuit of M has at least four elements.*
(b) *If $Z \subseteq C$, then*
(i) *if* $\delta(Z, E\setminus C)=1$, $|Z|\geqq 3$,
(ii) *if* $\delta(Z, E\setminus C)=2$, $|Z|=5$,
(iii) *if* $\delta(Z, E\setminus C)=3$, $|Z|\geqq 6$.

Omitting the relatively less involved verification that M is at least 3-connected, we suppose that (X, Y) is a partition of E with $|X|$, $|Y|\geqq 3$ and $r(X)+r(Y)=9$ (i.e. $\delta(X, Y)=2$). We consider two cases.
Case 1: $Y\subseteq C$ (so $X\supseteq E\setminus C$).
Case 2: $X\cap(E\setminus C)\neq\emptyset\neq Y\cap(E\setminus C)$.

In case 1 we cannot have $X=E\setminus C$, since this would imply $r(X)=4$, $r(Y)=6$. It follows that $r(Y)=7-r(X\cap C)$, $(=7-|X\cap C|)$, and

$$
\begin{aligned}
r(X)+r(Y) &= 4+r(X\cap C)-\delta(X\cap C, E\setminus C)+r(Y)\\
&= 11-\delta(X\cap C, E\setminus C).
\end{aligned}
$$

But then according to (b) (ii), $r(X)+r(Y)=9$ implies $|X\cap C|\geqq 5$, i.e. $|Y|\leqq 2$. This finishes case 1.

Now observe that by (a) neither X nor Y can contain a basis. Whe may therefore assume in case 2 that each of X, Y meets C. This yields

$$r(X)+r(Y) \geqq r(X\cap C)+1+r(Y\cap C)+1 = 9,$$

whence $r(X)=r(X\cap C)+1$, $r(Y)=r(Y\cap C)+1$. It follows that

$$\text{(c)} \qquad \delta(X\cap C, E\setminus C) \geqq \delta(X\cap C, X\cap(E\setminus C)) = \\ = r(X\cap C)+r(X\cap(E\setminus C))-r(X) = \\ = r(X\cap(E\setminus C))-1.$$

Now if $r(X\cap(E\setminus C))=4$, then (b)(iii) implies $r(X\cap C)=6$ and $r(X)=7$, a possibility we have already ruled out. From this (and the corresponding discussion for Y) we must conclude that $r(X\cap(E\setminus C))=r(Y\cap(E\setminus C))=3$. But then (c) and (b)(ii) imply that $|X\cap C|$ and $|Y\cap C|$ are each at least 5, providing our final contradiction.

3. Concluding remarks

First of all, one may ask whether there are other types of counterexamples to $(*)$, i.e. is it true that if M and X are a counterexample to $(*)$ with $k=4$, then M is obtained from some binary matroid M_0 on E by relaxing some circuit C with either $X\subseteq C$ or $X\subseteq E\setminus C$? (Here M_0 would necessarily be unique.)

But it seems to me preferable to regard $|X|=2$ $(=\text{rank}(U_4^2))$ as the natural level to which $(*)$ should be pushed. A loose rationale for this is as follows. It is an easy consequence of (2.3) of [3] that in a k-connected matroid any two flats of rank $t\leqq k-1$ have a common complement; and this immediately gives Seymour's Theorem (i.e. $k=3$ of $(*)$) in the special case that M has a U_4^2 minor of the form $M\setminus Z$. Similarly, if M is 4-connected and possesses some nonbinary rank 3 minor of the form $M\setminus Z$, then for any 3-element $X\subseteq E$ there is a rank 3 nonbinary minor which has X as a basis, and we may venture

Conjecture 1. *If M is a 4-connected nonbinary matroid and X a 3-element subset of E, then there is a rank 3 nonbinary minor of M having X as a basis.*

In fact, we expect that something extremely general can be said to the effect that information appearing somewhere in a matroid M must appear in many places if M is sufficiently connected. As an example of how this might be made concrete we offer

Conjecture 2. *For any matroid N there exist numbers $p(N)$ and $q(N)$ so that if M is a matroid of rank at least $p(N)$ and connectivity at least $q(N)$ and having N as a minor, and if F is any flat of M with $r(F)=r(N)$, then there is some modular complement J of F for which M/J has N as a minor.*

Or equivalently: any independent X of size $r(N)$ is the basis of some (rank $r(N)$) minor which has N as a minor.

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